

AN ALMOST SURE ENERGY INEQUALITY FOR MARKOV SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. We prove existence of weak martingale solutions satisfying an almost sure version of the energy inequality and which constitute a (almost sure) Markov process.

1. INTRODUCTION

Regardless several attempts, the well-posedness of the martingale problem for the stochastic (as well as the deterministic case) 3D Navier-Stokes equations remains an open problem (see [3], [11] for details on the martingale problem).

A major breakthrough has been the paper of Da Prato and Debussche [2] (see also [4], [8], [3]), where they show that there are *special* solutions which correspond to a Markov semigroup which is strong Feller and uniquely ergodic.

A different approach has been introduced in [7] (see also [5], [6], [10], [11]) where similar results have been proved with a completely different method. Here we follow this approach and consider the Navier-Stokes equations

$$(1.1) \quad \begin{cases} \dot{u} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = \dot{W}, \\ \operatorname{div} u = 0, \end{cases}$$

with periodic boundary conditions on the 3D torus.

The aim of this paper is to complete the work presented in [10], where it was proved that the semigroup associated to the Markov solutions introduced in [7] converges to a unique invariant measure. In order to prove that the rate of convergence is exponential, it was assumed the existence of Markov solutions satisfying an almost sure energy inequality. Such result is the main theorem (Theorem 2.3) of this paper (see also Remark 2.4). The method used here to prove the main result is essentially the same as in [1], where the almost sure energy balance was introduced to handle the space-time white noise forcing the equation.

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2. DEFINITIONS AND MAIN RESULT

We fix some notations we shall use throughout the paper and we refer to Temam [13] for a detailed account of all the definitions. Let $\mathbb{T}_3 = [0, 2\pi]^3$ and let \mathcal{D}^∞ be the space of infinitely differentiable divergence-free periodic vector fields $\varphi : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ with mean zero on \mathbb{T}_3 . Denote by H the closure of \mathcal{D}^∞ in $L^2(\mathbb{T}_3, \mathbf{R}^3)$ and by V the closure in $H^1(\mathbb{T}_3, \mathbf{R}^3)$. Denote by A , with domain $D(A)$, the *Stokes* operator and define the bi-linear operator $B : V \times V \rightarrow V'$ as the projection onto H of the nonlinearity of equation (1.1). Consider finally the abstract form of problem 1.1,

$$(2.1) \quad du + (\nu Au + B(u, u)) dt = Q^{\frac{1}{2}} dW,$$

where W is a cylindrical Wiener process on H and Q is a linear bounded symmetric positive operator on H with finite trace. Denote by $(e_k)_{k \in \mathbf{N}}$ a complete orthonormal system of eigenfunctions of Q , so that $Qe_k = \sigma_k^2 e_k$.

Next, we define the probabilistic framework where problem (2.1) is considered. Let $\Omega = C([0, \infty); D(A)')$, let \mathcal{B} be the Borel σ -field on Ω and let $\xi : \Omega \rightarrow D(A)'$ be the canonical process on Ω (that is, $\xi_t(\omega) = \omega(t)$). A filtration can be defined on \mathcal{B} as $\mathcal{B}_t = \sigma(\xi_s : 0 \leq s \leq t)$. Let $\Omega^t = C([t, \infty); D(A)')$ and denote by \mathcal{B}^t the Borel σ -field of Ω^t . Define the *forward shift* $\Phi_t : \Omega \rightarrow \Omega^t$ as $\Phi_t(\omega)(s) = \omega(s - t)$ for $s \geq t$. Given a probability P on (Ω, \mathcal{B}) , we shall denote by $\omega \mapsto P|_{\mathcal{B}_t}^\omega$, with $P|_{\mathcal{B}_t}^\omega \in \text{Pr}(\Omega^t)$, a *regular conditional probability distribution* of P , given \mathcal{B}_t .

For every $\varphi \in \mathcal{D}^\infty$ consider the process $(M_t^\varphi)_{t \geq 0}$ on Ω defined for $t \geq 0$ as

$$(2.2) \quad M_t^\varphi = \langle \xi_t - \xi_0, \varphi \rangle_H + \nu \int_0^t \langle \xi_s, A\varphi \rangle_H ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H ds.$$

Definition 2.1 (Weak martingale solution). Given $\mu_0 \in \text{Pr}(H)$, a probability P on (Ω, \mathcal{B}) is a *weak martingale solution* starting at μ_0 to problem (2.1) if

- [w1] $P[L_{\text{loc}}^2([0, \infty); H)] = 1$,
- [w2] for each $\varphi \in \mathcal{D}^\infty$ the process M_t^φ is square integrable and $(M_t^\varphi, \mathcal{B}_t, P)$ is a continuous martingale with quadratic variation $[M^\varphi]_t = t |Q^{\frac{1}{2}} \varphi|_H^2$,
- [w3] the marginal of P at time $t = 0$ is μ_0 .

Define for every $k \in \mathbf{N}$ the process $\beta_k(t) = \sigma_k^{-1} M_t^{e_k}$. Under a weak martingale solution, $(\beta_k)_{k \in \mathbf{N}}$ is a sequence of independent one dimensional Brownian motions. Thus, under any martingale solution, the process

$$(2.3) \quad W(t) = \sum_{k=0}^{\infty} \sigma_k \beta_k(t) e_k$$

is a Q -Wiener process and

$$(2.4) \quad z(t) = W(t) - \nu \int_0^t A e^{-\nu A(t-s)} W(s) ds$$

is the associated Ornstein-Uhlenbeck process starting at 0. Define the process $v(t, \cdot) = \xi_t(\cdot) - z(t, \cdot)$. Since $M_t^\Phi = \langle W(t), \Phi \rangle$ for every test function Φ , it follows that

$$\langle v(t) - \xi_0, \Phi \rangle_H + \nu \int_0^t \langle v(s), A\Phi \rangle_H ds - \int_0^t \langle B(\xi_s, \Phi), \xi_s \rangle_H ds = 0.$$

In different words, under a weak martingale solution P , v is a weak solution (i. e. a distributional solution) of the equation

$$(2.5) \quad \dot{v} + \nu Av + B(v + z, v + z) = 0, \quad P - \text{a. s.},$$

with initial condition $v(0) = \xi_0$. An energy balance functional can be associated to v , as it has been done in [7] for ξ ,

$$(2.6) \quad \mathcal{E}_t(v, z) = \frac{1}{2} |v_t|_H^2 + \nu \int_0^t |v_r|_V^2 dr - \int_0^t \langle z_r, B(v_r + z_r, v_r) \rangle dr.$$

Definition 2.2 (Energy martingale solutions). Given $\mu_0 \in \text{Pr}(H)$, a probability P on (Ω, \mathcal{B}) is a *energy martingale solution* starting at μ_0 of 2.1 if

- [e1] P is a weak martingale solution (see Definition 2.1),
- [e2] $P[v \in L_{\text{loc}}^\infty([0, \infty); H) \cap L_{\text{loc}}^2([0, \infty); V)] = 1$,
- [e3] there is a set $T_P \subset (0, \infty)$ of null Lebesgue measure such that for all $s \notin T_P$ and all $t \geq s$,

$$P[\mathcal{E}_t(v, z) \leq \mathcal{E}_s(v, z)] = 1.$$

Property [e3] is meaningful only if the map \mathcal{E} is finite, at least almost surely, with respect to a solution P , and measurable. By property [e2], v is P -a. s. weakly continuous in H and so $|v(t)|_H^2$ is defined point-wise in the energy estimate. Similarly, the other terms are also P -a. s. finite, again by [e2] and the regularity properties of z under P . Measurability is slightly more challenging and will be examined later in Proposition 3.2.

Theorem 2.3. *There exists a family $(P_x)_{x \in H}$ of energy martingale solutions such that the almost sure Markov property holds. More precisely, for every $x \in H$, for almost every $s \geq 0$ (including $s = 0$), for all $t \geq s$ and all bounded measurable $\phi : H \rightarrow \mathbf{R}$,*

$$(2.7) \quad \mathbb{E}^{P_x}[\phi(\xi'_t) | \mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\phi(\xi'_{t-s})].$$

The set of times where the Markov property fails to hold at some point x will be called the set of exceptional times of x .

Remark 2.4. Under appropriate assumptions on the covariance of the noise, Theorem 5.12 of [7] continues to hold thanks to property [e3]. In particular, the strong Feller property stated in [7, Theorem 5.11] and the unique ergodicity in [10].

As stated in [10], the proof detailed here, together with the corresponding proof of [7, theorem 4.1] ensure the existence of the *enhanced martingale solutions* used for the proof of exponential convergence to the invariant measure proved in [10].

3. PROOF OF THEOREM 2.3

Before proving the main theorem, it is preliminarily necessary to analyse more carefully the energy functional (2.6). Let $x \in H$, and let $z_x(t) = z(t) + e^{-vAt}x$ (that is, the solution to the Stokes problem starting at x). Similarly, set $v_x = \xi - z_x$.

Lemma 3.1. *Let P be an energy martingale solution. Then for every $x \in H$,*

$$P[\mathcal{E}_t(v_x, z_x) \leq \mathcal{E}_s(v_x, z_x)] = 1,$$

for a. e. $s \geq 0$ (including $s = 0$) and every $t \geq s$.

Proof. We just give a sketch of the proof (see for example [9] for a detailed proof in a more complicated case). Fix $x \in H$ and set $w(t) = e^{-vAt}x$. Then $z_x = z + w$ and $v_x = v - w$, hence $|v_x(t)|_H^2 = |v(t)|_H^2 + |w(t)|_H^2 - 2\langle v(t), w(t) \rangle_H$, and, since by assumptions the energy inequality holds for v , it is sufficient to prove a balance equality for w and $\langle v(t), w(t) \rangle_H$. The balance equality for w is straightforward by the PDE theory, so we only need to show that for all $s \geq 0$ and $t \geq s$,

$$\langle v_t, w_t \rangle_H + 2v \int_s^t \langle v_r, w_r \rangle_V dr + \int_s^t \langle v_r + z_r, B(v_r + z_r, w_r) \rangle dr = \langle v_s, z_s \rangle_H.$$

The above formula can be proved by standard methods, since by the regularity of w , $\langle v_t, w_t \rangle_H$ is differentiable in time and its derivative is $\langle \dot{v}, w \rangle + \langle v, \dot{w} \rangle$. By replacing \dot{v} with the corresponding terms in (2.5) and using the antisymmetric property of the nonlinearity, we get exactly the above formula. \square

Proposition 3.2. *Given $x \in H$, the map $(t, \omega) \mapsto \mathcal{E}_t(v_x(\omega), z_x(\omega))$, with $(t, \omega) \in [0, \infty) \times \Omega$, is progressively measurable and*

1. *for all $0 \leq s \leq t$, the sets $E_{s,t}(x) = \{\mathcal{E}_t(v_x, z_x) \leq \mathcal{E}_s(v_x, z_x)\}$ are \mathcal{B}_t -measurable;*
2. *for all $t > 0$, the sets*

$$E_t(x) = \{\mathcal{E}_t(v_x, z_x) \leq \mathcal{E}_s(v_x, z_x) \text{ for a. e. } s \leq t \text{ (including 0)}\}$$

are \mathcal{B}_t -measurable;

3. *the set*

$$E(x) = \mathcal{R} \cap \{\mathcal{E}_t(v_x, z_x) \leq \mathcal{E}_s(v_x, z_x) \text{ for a. e. } s \geq 0 \text{ (incl. 0), all } t \geq s\}$$

is \mathcal{B} -measurable, where

$$(3.1) \quad \mathcal{R} = \{z \in L_{loc}^8([0, \infty); L^4(\mathbb{T}_3)), v \in L_{loc}^\infty([0, \infty); H) \cap L_{loc}^2([0, \infty); V)\}.$$

Moreover, given P satisfying [e1] and [e2], property [e3] is equivalent to each of the following:

- [e3a] *There is $x \in H$ such that for each $t > 0$ there is a set $T \subset (0, t]$ of null Lebesgue measure and $P[E_{s,t}(x)] = 1$ for all $s \notin T$.*
- [e3b] *There is $x \in H$ such that for each $t > 0$, $P[E_t(x)] = 1$.*
- [e3c] *There is $x \in H$ such that $P[E(x)] = 1$.*

Proof. Measurability of the map \mathcal{E} follows from the semi-continuity properties of the various term of \mathcal{E} with respect to the topology of Ω . The measurability of each $E_{s,t}(x)$ now follows easily from measurability of \mathcal{E} . As it regards sets $E_t(x)$, fix $t > 0$ and notice that the Borel σ -algebra of the interval $(0, t)$ is countably generated, so that if \mathcal{T}_t is a countable basis,

$$E_t(x) = E_{0,t}(x) \cap \bigcap_{T \in \mathcal{T}_t} \left\{ \int_0^t \mathbb{1}_T(s) (\mathcal{E}_t(v_x, z_x) - \mathcal{E}_s(v_x, z_x)) ds \leq 0 \right\}$$

and all sets $\{\int_0^t \mathbb{1}_T(s) (\mathcal{E}_t(v_x, z_x) - \mathcal{E}_s(v_x, z_x)) ds \leq 0\}$ are \mathcal{B}_t -measurable by the measurability of \mathcal{E} . We next show measurability of $E(x)$. Let $J \subset [0, \infty)$ be a countable dense subset and define

$$\mathcal{R}_t = \{z \in L^8([0, t]; L^4(\mathbb{T}_3)), \quad v \in L^\infty(0, t; H) \cap L^2(0, t; V)\},$$

(notice that the regularity of z and v implies that of v_x and z_x), then $\mathcal{R}_t \in \mathcal{B}_t$ and, by the lower semi-continuity of the various terms of $\mathcal{E}_t(v_x, z_x) - \mathcal{E}_s(v_x, z_x)$ with respect to t , it follows that

$$E(x) = \bigcap_{t \in J} (\mathcal{R}_t \cap E_t(x))$$

is \mathcal{B} -measurable. The last statement of the lemma is now obvious from the above equalities, property **[e2]** and regularity of z . \square

We can proceed with the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. We use Theorem 2.8 of [7]. For every $x \in H$ let $\mathcal{C}(x)$ be the set of all energy martingale solutions starting at δ_x . It is sufficient to show that $(\mathcal{C}(x))_{x \in H}$ is an *a. s. pre-Markov family* (see [7, Definition 2.5]), namely,

1. each $\mathcal{C}(x)$ is non-empty, compact and convex, and $x \mapsto \mathcal{C}(x)$ is measurable (with respect to the Borel σ -field induced by the Hausdorff distance on compact sets).
2. For each $x \in H$ and each $P \in \mathcal{C}(x)$, $P[C([0, \infty); H_\sigma)] = 1$, where H_σ is the space H endowed with the weak topology.
3. For each $x \in H$ and $P \in \mathcal{C}(x)$ there is a set $T \subset (0, \infty)$ of null Lebesgue measure such that for all $t \notin T$ the following properties hold:
 - a. (*disintegration*) there is $N \in \mathcal{B}_t$ with $P(N) = 0$ such that $\omega \in H$ and $P|_{\mathcal{B}_t}^\omega \in \Phi_t \mathcal{C}(\omega(t))$ for all $\omega \notin N$,
 - b. (*reconstruction*) $P \otimes_t Q_\omega \in \mathcal{C}(x)$ for each \mathcal{B}_t -measurable map $\omega \mapsto Q_\omega$ with values in $\text{Pr}(\Omega^t)$ such that there is $N \in \mathcal{B}_t$ with $P(N) = 0$ and $\omega(t) \in H$, $Q_\omega \in \Phi_t \mathcal{C}(\omega(t))$ for all $\omega \notin N$,

where $P \otimes_t Q_\omega$ is the gluing of P and $\omega \mapsto Q_\omega$ (see [12, Chapter 6] for details). The first two properties are proved in Lemma 3.3, *disintegration* is proved in Lemma 3.4 and *reconstruction* is proved in Lemma 3.5. \square

Lemma 3.3. *For each $x \in H$ the set $\mathcal{C}(x)$ is non-empty compact convex and for all $P \in \mathcal{C}(x)$, $P[C([0, \infty); H_\sigma)] = 1$. Moreover, the map $x \mapsto \mathcal{C}(x)$ is Borel measurable.*

Proof. For the existence of weak martingale solutions see [7]. The proof of the energy inequality [e3] is indeed easier than the corresponding martingale property and can be carried on as in the deterministic case (see for example [13] or [9]). It is easy to show that $C(x)$ is convex, since all requirements of Definition 2.2 are linear with respect to measures $P \in \mathcal{C}(x)$. Finally, since under any $P \in \mathcal{C}(x)$ z is continuous with values in H , weak continuity of ξ follows from property [e2].

In order to prove both compactness and measurability it is sufficient to show that for each $x \in H$, for each sequence $(x_n)_{n \in \mathbb{N}}$ converging to x in H and for each $(P_n)_{n \in \mathbb{N}}$, with $P_n \in \mathcal{C}(x_n)$, the sequence $(P_n)_{n \in \mathbb{N}}$ has a limit point $P \in \mathcal{C}(x)$ (this follows from [12, Lemma 12.1.8]).

Let $x_n \rightarrow x$ in H and let $P_n \in \mathcal{C}(x_n)$. The energy inequality for each P_n ensures that $(P_n)_{n \in \mathbb{N}}$ is tight in $\Omega \cap L_{\text{loc}}^2([0, \infty); H)$, hence, up to a subsequence, $P_n \rightarrow P_\infty$ for some probability P_∞ . It remains to show that $P_\infty \in \mathcal{C}(x)$. Property [e1] can be proved essentially as in [7, Lemma 4.3] (without the complications of the super-martingale property).

Before proving [e2], [e3], we use Skorokhod theorem: there are a probability space $(\Sigma, \mathcal{F}, \mathbb{P})$ and random variables $(\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)_{n \in \mathbb{N}}$, $(\tilde{u}_\infty, \tilde{v}_\infty, \tilde{z}_\infty)$ such that each $(\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)$ has the same law of (ξ, v, z) under P_n for $0 \leq n \leq \infty$ and $\tilde{z}_n \rightarrow \tilde{z}_\infty$ in $L_{\text{loc}}^8([0, \infty); L^4(\mathbb{T}_3))$, $\tilde{v}_n \rightarrow \tilde{v}_\infty$ in $\Omega \cap L_{\text{loc}}^2([0, \infty); H)$ and weakly in $L_{\text{loc}}^2([0, \infty); V)$.

Property [e2] now follows by semicontinuity of the norms of spaces $L^\infty(0, T; H)$ and $L^2(0, T; V)$ with respect to the topology where $\tilde{v}_n \rightarrow \tilde{v}_\infty$, for all $T > 0$. In view of Proposition 3.2, we finally prove [e3a] (with $x = 0$). Fix $T > 0$, then since, by the proof of [e1], $\mathbb{E} \|\tilde{v}_n - \tilde{v}_\infty\|_{L^2(0, T; H)}^2 \rightarrow 0$, there is a null Lebesgue set $S \subset (0, T]$ such that for all $s \notin S$,

$$\mathbb{P}[\|\tilde{v}_{n'}(s)\|_H \rightarrow \|\tilde{v}_\infty(s)\|_H \text{ for a subsequence } (\tilde{v}_{n'})_{n' \in \mathbb{N}}] = 1.$$

Notice that $0 \notin S$, since we already know that $\tilde{v}_n(0) \rightarrow \tilde{v}_\infty(0)$. We are now able to prove [e3a] (with $x = 0$) for P_∞ . For each $n \in \mathbb{N}$ there is a null Lebesgue set $T_n \subset (0, T]$ such that $\mathbb{P}[\mathcal{E}_t(\tilde{v}_n, \tilde{z}_n) \leq \mathcal{E}_s(\tilde{v}_n, \tilde{z}_n)] = 1$, for all $s \notin T_n$. Let $T_\infty = S \cup \bigcup_n T_n$ and consider $s \notin T_\infty$. We have that $\mathcal{E}_t(\tilde{v}_n, \tilde{z}_n) \leq \mathcal{E}_s(\tilde{v}_n, \tilde{z}_n)$ for all $n \in \mathbb{N}$ and, in the limit as $n \rightarrow \infty$, by virtue of the convergence informations on \tilde{v}_n and \tilde{z}_n and of the semicontinuity properties of norms,

$$\mathcal{E}_t(\tilde{v}_\infty, \tilde{z}_\infty) - \mathcal{E}_s(\tilde{v}_\infty, \tilde{z}_\infty) \leq \liminf_n (\mathcal{E}_t(\tilde{v}_n, \tilde{z}_n) - \mathcal{E}_s(\tilde{v}_n, \tilde{z}_n)),$$

and in conclusion [e3a] is true. \square

Given $s \geq 0$ and $x \in H$, denote by $z(t|s, x)$ the Ornstein-Uhlenbeck process starting in x at time s , namely

$$z(t|s, x) = e^{-vA(t-s)}x + (W(t) - W(s)) - v \int_s^t A e^{-vA(t-r)}(W(r) - W(s)) dr.$$

In particular, $z(t|0, x) = z_x(t)$. Set moreover $v(t|s, x) = \xi - z(t|s, x)$. Given $t_0 > 0$, it is easy to see that for all $\omega \in \Omega^{t_0}$, $W(t, \Phi_{t_0}^{-1}(\omega)) = W(t + t_0, \omega) - W(t_0, \omega)$, and it depends

only on the values of ω in $[t_0, t_0 + t]$. Similarly,

$$(3.2) \quad \begin{aligned} z(t, \Phi_{t_0}^{-1}(\omega)|s, x) &= z(t + t_0, \omega|s + t_0, x), \\ v(t, \Phi_{t_0}^{-1}(\omega)|s, x) &= v(t + t_0, \omega|s + t_0, x). \end{aligned}$$

Lemma 3.4. *For every $x \in H$ and $P \in \mathcal{C}(x)$, there is a set $T \subset (0, \infty)$ of null Lebesgue measure, and for all $t \notin T$ there is $N \in \mathcal{B}_t$, with $P[N] = 0$, such that for all $\omega \notin N$, $\omega(t) \in H$ and $P|_{\mathcal{B}_t}^\omega \in \Phi_t \mathcal{C}(\omega(t))$.*

Proof. Fix $x \in H$ and $P \in \mathcal{C}(x)$, let T_P be the set of exceptional times of P and fix $t_0 \notin T_P$. We shall look for a P -null set $N \in \mathcal{B}_{t_0}$, with $N = N_1 \cup N_2 \cup N_3$, such that $\omega(t_0) \in H$ and $P|_{\mathcal{B}_{t_0}}^\omega \in \Phi_{t_0} \mathcal{C}(\omega(t_0))$.

The proof of property **[e1]** is the same as the proof of Lemma 4.4 of [7] and it provides a P -null set $N_1 \in \mathcal{B}_{t_0}$ out of which **[e1]** holds.

For any interval $J \subset [0, \infty)$, set $S_J = L_{\text{loc}}^\infty(J; H) \cap L_{\text{loc}}^2(J; V)$. In order to prove **[e2]**, we need to show that $P|_{\mathcal{B}_{t_0}}^\omega [v(\cdot|t_0, 0) \in S_{[t_0, \infty)}] = 1$. Set

$$(3.3) \quad \begin{aligned} S_{t_0} &= \{v \in S_{[0, t_0]}, e^{-vA(t-t_0)} z(t_0) \in S_{[0, \infty)}\}, \\ S^{t_0} &= \{v(\cdot|t_0, 0) \in S_{[t_0, \infty)}\}, \end{aligned}$$

then $S_{t_0} \in \mathcal{B}_{t_0}$, $S^{t_0} \in \mathcal{B}^{t_0}$ and, from property **[e2]**, it follows that $S_{t_0} \cap S^{t_0}$ is a P -full set, since $v(t|t_0, 0) = v(t) + e^{-vA(t-t_0)} z(t_0)$. By disintegration,

$$1 = P[S_{t_0} \cap S^{t_0}] = \mathbb{E}[\mathbb{1}_{S_{t_0}} P|_{\mathcal{B}_{t_0}}^\omega [S^{t_0}]],$$

hence there is a P -null set $N_2 \in \mathcal{B}_{t_0}$ such that $P|_{\mathcal{B}_{t_0}}^\omega [S^{t_0}] = 1$ for all $\omega \notin N_2$.

Finally, we prove that **[e3c]** holds for conditional probabilities. Let \mathcal{R}_{t_0} be defined as \mathcal{R} in (3.1) but on the time interval $[0, t_0]$, and set

$$\begin{aligned} A &= \mathcal{R} \cap \{\mathcal{E}_t(v, z) \leq \mathcal{E}_s(v, z) \text{ for a. e. } s \geq 0 \text{ (including } 0, t_0), \text{ all } t \geq s\} \\ A_{t_0} &= \mathcal{R}_{t_0} \cap \{\mathcal{E}_t(v, z) \leq \mathcal{E}_s(v, z) \text{ for a. e. } s \in [0, t_0] \text{ (incl. } 0, t_0), \text{ all } t \in [s, t_0]\}, \end{aligned}$$

We have that $A_{t_0} \in \mathcal{B}_{t_0}$ and, since $t_0 \notin T_P$, $P[A] = P[A_{t_0}] = 1$. Now, if $\bar{\omega} \in A_{t_0}$, set $B(\bar{\omega}) = A \cap \{\omega : \omega = \bar{\omega} \text{ on } [0, t_0]\}$, then $B(\bar{\omega})$ is equal to

$$\{\mathcal{E}_t(v^{\bar{\omega}}, z^{\bar{\omega}}) \leq \mathcal{E}_s(v^{\bar{\omega}}, z^{\bar{\omega}}) \text{ for a. e. } s \geq t_0 \text{ (including } t_0), \text{ all } t \geq s\},$$

since $v(t + t_0, \omega) = v(t + t_0, \omega|t_0, z(t_0, \omega))$ (and similarly for z), and we have set $v^{\bar{\omega}}(\cdot) = v(\cdot|t_0, z(t_0, \bar{\omega}))$ and $z^{\bar{\omega}}(\cdot) = z(\cdot|t_0, z(t_0, \bar{\omega}))$. Moreover, the map $\omega \mapsto \mathbb{1}_{A_{t_0}}(\omega) P|_{\mathcal{B}_{t_0}}^\omega [B(\omega)]$ is \mathcal{B}_{t_0} -measurable, since $P|_{\mathcal{B}_{t_0}}^\omega [B(\omega)] = P|_{\mathcal{B}_{t_0}}^\omega [A]$. By **[e3c]** (with $x = 0$) for P , and disintegration, $1 = P[A] = \mathbb{E}^P[\mathbb{1}_{A_{t_0}}(\cdot) P|_{\mathcal{B}_{t_0}}^\omega [B(\cdot)]]$, and so there is $N_3 \in \mathcal{B}_{t_0}$ such that $P|_{\mathcal{B}_{t_0}}^\omega [B(\omega)] = 1$ for all $\omega \notin N_3$, hence **[e3c]** holds, with $x = z(t_0, \omega)$. \square

Lemma 3.5. *For every $x \in H$ and $P \in \mathcal{C}(x)$, there is a set $T \subset (0, \infty)$ of null Lebesgue measure such that $P \otimes_t Q \in \Phi_t \mathcal{C}(\omega(t))$ for all $t \notin T$. The statement must hold for each*

\mathcal{B}_t -measurable map $\omega \mapsto Q_\omega$ with values in $\text{Pr}(\Omega^t)$ such that there is $N_Q \in \mathcal{B}_t$ with $P[N_Q] = 0$, and $\omega(t) \in H$, $Q_\omega \in \Phi_t \mathcal{C}(\omega(t))$, for all $\omega \notin N_Q$.

Proof. Let $x \in H$, $P \in \mathcal{C}(x)$ and let T_P be the set of exceptional times of P . Fix $t_0 \notin T_P$ and let $\omega \mapsto Q_\omega$ according to the statement of the lemma. Everything boils down to show that $P \otimes_{t_0} Q_\cdot \in \mathcal{C}(x)$, and, as in the proof of the previous lemma, we refer to [7] (see Lemma 4.5) for the proof of [e1].

To verify [e2], consider again sets S_{t_0} and S^{t_0} defined in (3.3). By [e2] for Q_\cdot , for each $\omega \notin N_Q$, $Q_\omega[S^{t_0}] = 1$. Moreover, by [e2] for P , it follows that $P[S_{t_0}] = 1$. Finally, since we know that $v(t + t_0, \omega) = v(t + t_0, \omega|_{t_0}, 0) - e^{-vAt} z(t_0, \omega)$, it follows easily that $S_{t_0} \cap S^{t_0} = \{v \in S_{[0, \infty)}\}$ and so

$$(P \otimes_{t_0} Q_\cdot)[v \in S_{[0, \infty)}] = (P \otimes_{t_0} Q_\cdot)[S_{t_0} \cap \mathcal{R}^{t_0}] = \mathbb{E}^P[\mathbb{1}_{S_{t_0}} Q_\omega[S^{t_0}]] = 1.$$

Finally, we prove [e3c]. Define A and A_{t_0} as in the proof of the previous lemma. Since $t_0 \notin T_P$ and $A_{t_0} \in \mathcal{B}_{t_0}$, we know that $(P \otimes_{t_0} Q_\cdot)[A_{t_0}] = P[A_{t_0}] = 1$. Define $B(\bar{\omega}) = A \cap \{\omega : \omega = \bar{\omega} \text{ on } [0, t_0]\}$ and notice that, if $\omega \in A_{t_0} \cap N_Q^c$ (which is again a \mathcal{B}_{t_0} -measurable $(P \otimes_{t_0} Q_\cdot)$ -full set), then by [e3c] (with $x = z(t_0, \omega)$) for Q_ω it follows that $Q_\omega[B(\omega)] = 1$. The map $\omega \mapsto \mathbb{1}_{A_{t_0} \cap N_Q^c}(\omega) Q_\omega[B(\omega)]$ is then trivially \mathcal{B}_{t_0} -measurable and equal to 1, P -a.s.. Moreover, $Q_\omega[A] = Q_\omega[B(\omega)] = 1$ for all $\omega \in A_{t_0} \cap N_Q^c$ and so

$$(P \otimes_{t_0} Q_\cdot)[A] = \mathbb{E}^P[\mathbb{1}_{A_{t_0} \cap N_Q^c} Q_\cdot[B(\cdot)]] = P[A_{t_0} \cap N_Q^c] = 1,$$

in conclusion, [e3c] (with $x = 0$) holds for $P \otimes_{t_0} Q_\cdot$. □

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